

Final Exam — Functional Analysis (WIFA–08)

Tuesday 4 April 2017, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
 2. All answers need to be accompanied with an explanation or a calculation: only answering “yes”, “no”, or “42” is not sufficient.
 3. If p is the number of marks then the exam grade is $G = 1 + p/10$.
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Problem 1 (10 + 5 + 5 + 5 = 25 points)

Consider the following normed linear space:

$$X = \{f : [a, b] \rightarrow \mathbb{K} : f \text{ is bounded}\},$$
$$\|f\| = \sup_{x \in [a, b]} |f(x)|.$$

- (a) Prove that X is a Banach space (i.e., every Cauchy sequence has a limit).
- (b) Prove that $V = \{f \in X : f(a) = f(b) = 0\}$ is a linear subspace of X .
- (c) Prove that V is closed in X .
- (d) Compute $\dim(X/V)$.

Problem 2 (6 + 3 + 8 + 8 = 25 points)

Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| < 1$ and consider the following linear operator:

$$T : \ell^2 \rightarrow \ell^2, \quad (x_1, x_2, x_3, \dots) \mapsto (\alpha x_1, \alpha^2 x_2, \alpha^3 x_3, \dots).$$

Prove the following statements:

- (a) $\|T\| = |\alpha|$;
- (b) T is selfadjoint if and only if $\alpha \in \mathbb{R}$;
- (c) T is compact;
- (d) $\sigma(T) = \{\alpha^n : n \in \mathbb{N}\} \cup \{0\}$.

Problem 3 (5 + 3 + 7 + 5 = 20 points)

- (a) Formulate Baire’s theorem for metric spaces.
- (b) Let $\|\cdot\|$ be any norm on the space

$$\mathcal{P} = \{p : \mathbb{K} \rightarrow \mathbb{K} : p \text{ is a polynomial}\}.$$

Prove the following statements:

- (i) $\mathcal{P}_n = \{p \in \mathcal{P} : \deg p \leq n\}$ is closed for each $n \in \mathbb{N} \cup \{0\}$;
- (ii) \mathcal{P}_n is nowhere dense for each $n \in \mathbb{N} \cup \{0\}$;
- (iii) \mathcal{P} is *not* a Banach space.

Problem 4 (5 + 5 + 3 + 7 = 20 points)

- (a) Formulate the Hahn-Banach theorem for normed linear spaces.
- (b) Let X be an infinite-dimensional normed linear space over \mathbb{C} . Pick $x_0 \in X$, $x_0 \neq 0$, and let $V = \text{span}\{x_0\}$. Define the linear functional $f : V \rightarrow \mathbb{C}$ by setting $f(\lambda x_0) = (1 + 4i)\lambda\|x_0\|$.

Does there exist a functional $g \in X'$ such that $g|_V = f$ and:

- (i) $\|g\| = 4$?
- (ii) $\|g\| = \sqrt{17}$?
- (iii) $\|g\| \geq 23$?

End of test (90 points)

Solution of Problem 1 (10 + 5 + 5 + 5 = 25 points)

- (a) If (f_n) is a Cauchy sequence in X , then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \quad \Rightarrow \quad \|f_n - f_m\| \leq \varepsilon.$$

In particular, for each $x_0 \in [a, b]$ it follows that

$$n, m \geq N \quad \Rightarrow \quad |f_n(x_0) - f_m(x_0)| \leq \varepsilon, \quad (1)$$

which means that $(f_n(x_0))$ is a Cauchy sequence in \mathbb{K} .

(3 points)

Since \mathbb{K} is complete the sequence $(f_n(x_0))$ converges. Hence, we can define $f : [a, b] \rightarrow \mathbb{K}$ by

$$f(x_0) = \lim_{n \rightarrow \infty} f_n(x_0), \quad x_0 \in [a, b].$$

(2 points)

Taking $m \rightarrow \infty$ in the inequality (1) gives

$$n \geq N \quad \Rightarrow \quad |f_n(x_0) - f(x_0)| \leq \varepsilon,$$

and since $x_0 \in [a, b]$ is arbitrary it follows that

$$n \geq N \quad \Rightarrow \quad \|f_n - f\| = \sup_{x_0 \in [a, b]} |f_n(x_0) - f(x_0)| \leq \varepsilon,$$

which means that $f_n \rightarrow f$ in X .

(3 points)

Finally, with $n = N$ it follows that $f_N - f \in X$ so that $f = f_N - (f_N - f) \in X$.

(2 points)

- (b) If $f, g \in V$ and $\lambda, \mu \in \mathbb{K}$ then

$$(\lambda f + \mu g)(a) = \lambda f(a) + \mu g(a) = 0,$$

$$(\lambda f + \mu g)(b) = \lambda f(b) + \mu g(b) = 0,$$

which implies that $\lambda f + \mu g \in V$. This proves that V is a linear subspace of X .

(5 points)

- (c) If $f \in \overline{V}$, then there exists a sequence $f_n \in V$ such that $f_n \rightarrow f$. Hence

$$|f(a)| = |f(a) - f_n(a)| \leq \|f - f_n\| \rightarrow 0,$$

which implies that $f(a) = 0$. Similarly, it follows that $f(b) = 0$. We conclude that $f \in V$ so that V is closed.

(5 points)

- (d) Define the linear map $T : X \rightarrow \mathbb{K}^2$ by $Tf = (f(a), f(b))$. Then $\ker T = V$ and obviously $\text{ran } T = \mathbb{K}^2$. Note that $X/\ker T \simeq \text{ran } T$ which implies that $\dim(X/V) = 2$.

(5 points)

Solution of Problem 2 (6 + 3 + 8 + 8 = 25 points)

- (a) Since $|\alpha| < 1$ it follows that $|\alpha|^n \leq |\alpha|$ for each $n \in \mathbb{N}$. Let $x \in \ell^2$ be arbitrary, then

$$\|Tx\|^2 = \sum_{n=1}^{\infty} |\alpha^n x_n|^2 = \sum_{n=1}^{\infty} |\alpha|^{2n} |x_n|^2 \leq |\alpha|^2 \sum_{n=1}^{\infty} |x_n|^2 = |\alpha|^2 \|x\|^2,$$

which shows that

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} \leq |\alpha|.$$

(4 points)

Note that for $x = (1, 0, 0, \dots)$ we have $\|x\| = 1$ and $\|Tx\| = |\alpha|$ which implies that $\|T\| = |\alpha|$.

(2 points)

- (b) If $x, y \in \ell^2$ then

$$(Tx, y) = \sum_{n=1}^{\infty} \alpha^n x_n \bar{y}_n = \sum_{n=1}^{\infty} x_n \overline{\alpha^n y_n} = (x, T^*y)$$

which shows that $T^*y = (\bar{\alpha}y_1, \bar{\alpha}^2y_2, \bar{\alpha}^3y_3, \dots)$. In particular, it follows that $T = T^*$ if and only if $\alpha \in \mathbb{R}$.

(3 points)

- (c) Define for $k \in \mathbb{N}$ the operator

$$T_k : \ell^2 \rightarrow \ell^2, \quad (x_1, x_2, x_3, \dots) \mapsto (\alpha x_1, \dots, \alpha^k x_k, 0, 0, 0, \dots)$$

The same argument as in part (a) shows that T_k is bounded. In addition, $\text{ran } T_k$ is finite-dimensional, which implies that T_k is compact.

(4 points)

Let $x \in \ell^2$ be arbitrary, then

$$\|(T - T_k)x\|^2 = \sum_{n=k+1}^{\infty} |\alpha|^{2n} |x_n|^2 \leq |\alpha|^{2k+2} \sum_{n=k+1}^{\infty} |x_n|^2 \leq |\alpha|^{2k+2} \|x\|^2.$$

which shows that $\|T - T_k\| \leq |\alpha|^{2k+2} \rightarrow 0$ as $k \rightarrow \infty$. Since each T_k is compact it follows that T is compact as well.

(4 points)

- (d) Clearly, α^n is an eigenvalue of T for each $n \in \mathbb{N}$. The corresponding eigenvector is given by the n -th standard unit vector. Hence, $\{\alpha^n : n \in \mathbb{N}\} \subset \sigma(T)$.

(2 points)

Note that $\alpha^n \rightarrow 0$ since $|\alpha| < 1$. Since the spectrum is closed it follows that $0 \in \sigma(T)$ as well.

(1 point)

If $\lambda \notin \{\alpha^n : n \in \mathbb{N}\} \cup \{0\}$, then there exists $\delta > 0$ such that $|\lambda - \alpha^n| \geq \delta$ for all $n \in \mathbb{N}$. Note that

$$(T - \lambda)^{-1}x = \left(\frac{x_1}{\alpha - \lambda}, \frac{x_2}{\alpha^2 - \lambda}, \frac{x_3}{\alpha^3 - \lambda}, \dots \right)$$

so that

$$\|(T - \lambda)^{-1}x\|^2 = \sum_{n=1}^{\infty} \frac{|x_n|^2}{|\alpha^n - \lambda|^2} \leq \frac{1}{\delta^2} \sum_{n=1}^{\infty} |x_n|^2 = \frac{1}{\delta^2} \|x\|^2,$$

which shows that $(T - \lambda)^{-1}$ is bounded so that $\lambda \in \rho(T)$. Hence, $\sigma(T) = \{\alpha^n : n \in \mathbb{N}\} \cup \{0\}$.

(5 points)

Solution of Problem 3 (5 + 3 + 7 + 5 = 20 points)

(a) Let X be a complete metric space and let $O \subset X$ be nonempty and open. Then O is nonmeager.

(5 points)

(b) (i) Note that $\mathcal{P}_n = \{p \in \mathcal{P} : \deg p \leq n\}$ is a finite-dimensional subspace of the normed linear space \mathcal{P} . This implies that \mathcal{P}_n is closed.

(3 points)

(ii) We need to prove that $\text{int } \overline{\mathcal{P}_n} = \emptyset$, or, equivalently, since \mathcal{P}_n is closed, that $\text{int } \mathcal{P}_n = \emptyset$.

(2 points)

If $p_0 \in \text{int } \mathcal{P}_n$ then there exists $\varepsilon > 0$ such that

$$\{p \in \mathcal{P} : \|p - p_0\| < \varepsilon\} \subset \mathcal{P}_n.$$

Let $q \in \mathcal{P}$ be a nonzero polynomial and define $\tilde{q} = p_0 + \frac{1}{2}\varepsilon q / \|q\|$ then

$$\|\tilde{q} - p_0\| = \frac{1}{2}\varepsilon,$$

which implies that $\tilde{q} \in \mathcal{P}_n$. In turn, this implies that

$$q = \frac{2\|q\|}{\varepsilon}(\tilde{q} - p_0) \in \mathcal{P}_n$$

$q \in \mathcal{P}_n$ so that $\mathcal{P} = \mathcal{P}_n$, which is a contradiction. Hence, $\text{int } \mathcal{P}_n = \emptyset$.

(5 points)

(iii) If \mathcal{P} is a Banach space, then it is also a complete metric space. Since

$$\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$$

it would follow from Baire's theorem that at least one of the sets \mathcal{P}_n is *not* nowhere dense. This contradicts the conclusion of part (ii). Hence, we conclude that \mathcal{P} is *not* a Banach space.

(5 points)

Solution of Problem 4 (5 + 5 + 3 + 7 = 20 points)

- (a) Let X be a normed linear space and $V \subset X$ a linear subspace. For each $f \in V'$ there exists $F \in X'$ such that $F \upharpoonright V = f$ and $\|F\| = \|f\|$.

(5 points)

- (b) (i) Note that

$$\|f\| = \sup_{\lambda \neq 0} \frac{|f(\lambda x_0)|}{\|\lambda x_0\|} = \sup_{\lambda \neq 0} \frac{|1 + 4i| |\lambda| \|x_0\|}{|\lambda| \|x_0\|} = |1 + 4i| = \sqrt{17}.$$

This means that if $g \in X'$ and $g \upharpoonright V = f$ then $\|g\| \geq \|f\| = \sqrt{17}$. This implies that there does not exist an extension g of f such that $\|g\| = 4$.

(5 points)

- (ii) By the Hahn-Banach theorem there exists an extension $g \in X'$ of f such that $\|g\| = \|f\| = \sqrt{17}$.

(3 points)

- (iii) Pick a nontrivial $x_1 \in X$ such that $\{x_0, x_1\}$ is a linearly independent set and define $g : \text{span}\{x_0, x_1\} \rightarrow \mathbb{C}$ by setting

$$g(\lambda x_0 + \mu x_1) = (1 + 4i)\lambda \|x_0\| + 23\mu \|x_1\|. \quad (2)$$

Clearly, g is linear and $g \upharpoonright \text{span}\{x_0\} = f$. Since g is defined on a *finite-dimensional* space it is automatically bounded. Note that $|g(\mu x_1)| / \|\mu x_1\| = 23$ for all $\mu \neq 0$ so that $\|g\| \geq 23$.

(5 points)

Now use the Hahn-Banach theorem to extend g to all of X while preserving the norm.

(2 points)